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On the crack-tip stress singularity of interfacial cracks in transversely isotropic piezoelectric bimaterials

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Abstract

In this paper, characteristics of the interface crack-tip stress and electric displacement fields in transversely isotropic piezoelectric bimaterials are studied. The authors have proven, within the framework of the generalized Stroh formalism for piezoelectric bimaterials, that there is no coexistence of the parameters ε (oscillating) and κ (non-oscillating) in the interface crack-tip generalized stress field for all transversely isotropic piezoelectric bimaterials. This leads to the classification of piezoelectric bimaterials into one group that exhibits the oscillating property in the interface crack-tip generalized stress field and the other that does not. Fifteen (15) pair-combinations of six (6) piezoelectric materials PZT-4, PZT-5H, PZT-6B, PZT-7A, P-7, and $BaTiO_3$, which are commonly used in practice, are numerically analyzed in this study, and the results backup the above theoretical conclusions. Moreover, the associated eigenvectors for such material systems (with either $\varepsilon = 0$ or $\kappa = 0$) are also obtained numerically, and the result show that there still exist four linear independent associate eigenvectors for each bimaterial.

Keywords: Transversely isotropic piezoelectric material; Interface crack; Stroh formula; Dissimilar piezoelectric material; Fracture

1. Introduction

For piezoelectric materials, it has been shown by Suo et al. (1992) that the crack-tip stress and electric displacement field singularity at the bimaterial interface has a characteristic of $r^{-1/2+i\gamma}$, where γ takes four eigenvalues: two are real ($\gamma_{1,2} = \pm\varepsilon$) and two are imaginary ($\gamma_{3,4} = \pm i\kappa$). Thus the fracture behavior of the interfacial crack in such a bimaterial system is governed by two numbers, ε and κ, with ε controlling its

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oscillatory singularity and κ modifying its non-oscillatory singularity, as oppose to that ($r^{-1/2}$) of the crack in a monolithic homogeneous material. In the treatment by Suo et al. (1992), both non-zero values of ε and κ were assumed and hence four linearly independent eigenvectors would be obtained, associated with the four eigenvalues, for an impermeable interface crack in a piezoelectric bimaterial. Other researchers, for example, Boem and Atluri (1996) and Ma and Chen (2001), have extended the work on similar basis.

A reduced class from the generalized piezoelectric bimaterials is the class of transversely isotropic piezoelectric bimaterials, which has more practical significance, because almost all piezoelectric materials that are in use today fall into this category. The crack-tip singular field characteristics, in regard to the existence of (non-zero) ε and κ, have not been studied in sufficient detail for this class of materials. Up to date, particularly, none of ε and κ values for practical piezoelectric materials have been reported in the literature.

There have been some discussions for the condition where the crack front coincides with the poling axis (Deng and Meguid, 1999a,b; Wang and Zhong, 2002). In this case, the problem can be decoupled into: (a) an in-plane problem and (b) an anti-plane problem, the latter problem has been studied by Deng and Meguid (1999a,b) and they found that the mode-III stress and electric fields exhibit the traditional inverse square root singularity in any transversely isotropic piezoelectric bimaterials. However, the more important case where the crack is perpendicular to the poling axis has not been discussed in sufficient detail. The question of how the piezoelectric poling effect is affected by the existence of an impermeable crack is of most engineering concern, which is the focus of this study.

The evaluation of ε and κ is based on the material matrix $\mathbf{H}$, as proposed by Suo et al. (1992), which is constructed by solving the following eigenvalue problem for the material:

$$[\mathbf{Q} + p(\mathbf{R} + \mathbf{R}^{\mathrm{T}}) + p^2\mathbf{T}]\mathbf{a} = \mathbf{0},$$

according to Stroh's formulism (Stroh, 1958; Ting, 1986, 1990; Suo, 1990) for piezoelectric materials (Suo et al., 1992; Boem and Atluri, 1996, Deng and Meguid, 1998; Ma and Chen, 2001), where $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{T}$ are material matrices constructed with the material's elastic constants, piezoelectric constants and dielectric constants (see Section 2 for detail).

In this paper, we find, by showing that the determinant of the imaginary part of $\mathbf{H}$ always vanishes for any given transversely isotropic piezoelectric bimaterial system, that ε and κ cannot be both non-zero, that is, either the condition $\varepsilon = 0$ or $\kappa = 0$ must exist in such a material system. This theoretical conclusion is then validated by evaluations for practical piezoelectric materials such as PZT-4, PZT-5H, $BaTiO_3$, PZT-6B, PZT-7A and P-7. This theoretical finding has two significant implications: (I) non-oscillating stress singularity may exist at the interfacial crack-tip in some transversely isotropic piezoelectric bimaterials, while in some others with $\kappa = 0$ and $\varepsilon \neq 0$ oscillating crack-tip stress field prevails; (II) the non-coexistence condition for ε and κ creates a special case where some eigenvectors are associated with eigenvalues of zero. The first implication (I) can be used to classify piezoelectric materials in the study of their fracture behaviors. The second implication (II) still needs further theoretical investigation, which is the subject of another treatment underway. Numerical results for some practical piezoelectric bimaterials are given in this paper to show that there still exist four linear independent associate eigenvectors with such values of ε and κ. Since most of the practical piezoelectric materials are transversely isotropic materials in nature, it also renders practical importance to consider the above two implications.

To help the readers to follow the theoretical derivation and reach the conclusion, the basic Stroh formulism and the solution given by Suo et al. (1992) are briefly described in Section 2, and then in Section 3, the explicit solution of the eigenvalues for transversely isotropic piezoelectric materials is obtained, in which the determinant of the imaginary part of $\mathbf{H}$ always vanishes for all transversely isotropic piezoelectric bimaterials. The values of ε and κ for some piezoelectric bimaterials are given to support the theoretical finding.

2. The basic formula

The generalized displacements, $\mathbf{u}$ (displacements u_i and the electric potential ϕ), and stresses, $\boldsymbol{\sigma}$ (stresses σ_{i2} and the resultant force D_2), in-plane problems of linear piezoelectric materials can be expressed in terms of the four-dimensional Stroh formalism as follows (Suo et al., 1992; Boem and Atluri, 1996; Deng and Meguid, 1998; Ma and Chen, 2001):

$$\mathbf{u} = 2\,\mathrm{Re}[\mathbf{A}\mathbf{f}(z)], \quad [\sigma_{i2}] = 2\,\mathrm{Re}[\mathbf{B}\mathbf{f}'(z)], \tag{1}$$

where

$$\mathbf{u} = (u_1, u_2, u_3, \phi)^{\mathrm{T}}, \quad [\sigma_{i2}] = (\sigma_{12}, \sigma_{22}, \sigma_{32}, D_2)^{\mathrm{T}}, \tag{2}$$

and $\mathbf{f}(z)$ is a column of four complex potential functions, as

$$\begin{aligned} &\mathbf{f}(z) = (f_1(z_1), f_2(z_2), f_3(z_3), f_4(z_4))^{\mathrm{T}}, \\ &z_\alpha = x_1 + p_\alpha x_2 \quad (\alpha = 1, 2, 3, 4), \end{aligned} \tag{3}$$

where p_α are the eigenvalues with positive imaginary parts of the following material's characteristic equation (equilibrium condition):

$$[\mathbf{Q} + p(\mathbf{R} + \mathbf{R}^{\mathrm{T}}) + p^2\mathbf{T}]\mathbf{a} = \mathbf{0}, \tag{4}$$

with the material matrices $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{T}$ constructed as

$$\mathbf{Q} = \begin{bmatrix} c_{i1k1} & e_{1i1} \\ e_{1k1}^{\mathrm{T}} & -\chi_{11} \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} c_{i1k2} & e_{2i1} \\ e_{1k2}^{\mathrm{T}} & -\chi_{12} \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} c_{i2k2} & e_{2i2} \\ e_{2k2}^{\mathrm{T}} & -\chi_{22} \end{bmatrix}, \tag{5}$$

where c_{ijkl}, e_{kij}, and χ_{ik} denote the elasticity constants, the piezoelectric constants and the dielectric constants, respectively.

Then, matrices $\mathbf{A}$ and $\mathbf{B}$ can be constructed with the corresponding eigenvectors, as

$$\mathbf{A} = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4), \quad \mathbf{B} = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4), \tag{6}$$

$$\mathbf{b}_j = (\mathbf{R}^{\mathrm{T}} + p_j\mathbf{T})\mathbf{a}_j = -(1/p_j)(\mathbf{Q} + p_j\mathbf{R})\mathbf{a}_j \quad (j = 1, 2, 3, 4). \tag{7}$$

As can be seen from Eqs. (1)–(7), the electric potential and the electric displacement have been brought into the formulation as components of the generalized displacement and generalized stress, respectively, extending the formalism to a four-dimension problem. The formalism compacts the elastic–piezoelectric interactions nicely into a unified formalism that is consistent with the treatment of anisotropic elasticity, which makes the problem solving manipulations easier as shown in the following.

When an impermeable crack exist at the interface between two piezoelectric materials, two Hermitian matrices, $\mathbf{Y}$ and $\mathbf{H}$, can be defined for the problem, as (Suo et al., 1992)

$$\mathbf{Y} = \mathrm{i}\mathbf{A}\mathbf{B}^{-1}, \quad \mathbf{H} = \mathbf{Y}_1 + \overline{\mathbf{Y}}_2, \tag{8}$$

where the subscripts "1" and "2" denotes the different materials.

By satisfying the continuity conditions of the generalized displacement and traction at the interface, a column vector function $\mathbf{h}(z)$ can be defined, as (Suo et al., 1992)

$$\mathbf{h}(z) = \begin{cases} \mathbf{B}_1\mathbf{f}_1'(z) & y > 0, \\ \mathbf{H}^{-1}\overline{\mathbf{H}}\mathbf{B}_2\mathbf{f}_2'(z) & y < 0, \end{cases} \tag{9}$$

which is analytic in the whole plane except on the cracking line. Once $\mathbf{h}(z)$ is found, the whole problem is solved.

Using the above notions, the condition of stress free and electric displacement free on the crack surface yields a homogeneous Hilbert problem as expressed in the following form:

$$\mathbf{h}^{+}(x) + \overline{\mathbf{H}}^{-1}\mathbf{H}\mathbf{h}^{-}(x) = 0. \tag{10}$$

Assuming that the function $\mathbf{h}(z)$ takes the form:

$$\mathbf{h}(z) = \mathbf{w}z^{-1/2+\mathrm{i}\gamma}, \tag{11}$$

where $\mathbf{w}$ is a four-element column vector and γ an arbitrary number, both to be determined, Eq. (10) turns into the following eigenvalue problem:

$$\overline{\mathbf{H}}\mathbf{w} = e^{2\pi\gamma}\mathbf{H}\mathbf{w}. \tag{12}$$

Separating the matrix $\mathbf{H}$ into a real part $\mathbf{D}$ and an imaginary $\mathbf{W}$, as

$$\mathbf{H} = \mathbf{D} + \mathrm{i}\mathbf{W}. \tag{13}$$

Eq. (12) leads to

$$\|\mathbf{D}^{-1}\mathbf{W} + \mathrm{i}\beta\mathbf{I}\| = \beta^4 + 2b\beta^2 + c = 0, \tag{14}$$

where

$$\gamma = -\frac{1}{\pi}\tanh^{-1}(\beta), \quad b = \frac{1}{4}tr[(\mathbf{D}^{-1}\mathbf{W})^2], \quad c = \|\mathbf{D}^{-1}\mathbf{W}\|. \tag{15}$$

β is the root of Eq. (14) and hence γ may take four distinct values $\pm\varepsilon$, and $\pm\mathrm{i}\kappa$ as obtained by Suo et al. (1992) and Boem and Atluri (1996) for a general case,

$$\varepsilon = \frac{1}{\pi}\tanh^{-1}[(b^2-c)^{1/2} - b]^{1/2}, \quad \kappa = \frac{1}{\pi}\tanh^{-1}[(b^2-c)^{1/2} + b]^{1/2}. \tag{16}$$

With the four distinct (non-zero) eigenvalues, four associated linearly independent eigenvectors $\mathbf{w}_1$, $\overline{\mathbf{w}}_1$, $\mathbf{w}_3$, $\mathbf{w}_4$ can be obtained accordingly and thus the solution of the problem is obtained, as represented by Eqs. (1), (9) and (11). The parameters ε and κ control the oscillatory (or non-oscillatory) singularity behavior of the generalized crack-tip stress field. The solution of this form has been used by many other researchers in their investigation on piezoelectric materials as did by Boem and Atluri (1996) and Ma and Chen (2001).

3. Proof of the non-coexistence of ε and κ

It may seem to be straightforward to calculate the value of ε and κ from Eq. (16) for any anisotropic piezoelectric bimaterial system. However, this is not the case for transversely isotropic piezoelectric bimaterials. Why? It is all because for transversely isotropic piezoelectric bimaterials

$$\|\mathbf{W}\| = 0 \quad \text{and hence} \quad c = 0, \tag{17}$$

as the authors shall prove in the following derivations.

For a transversely isotropic piezoelectric material, taking x_3 to be parallel to the poling axis of the material, by convention, the material's constitutive relation is expressed in the following form:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{c_{11}-c_{12}}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \\ 2\varepsilon_{12} \end{Bmatrix} - \begin{bmatrix} 0 & 0 & e_{13} \\ 0 & 0 & e_{13} \\ 0 & 0 & e_{33} \\ 0 & e_{15} & 0 \\ e_{15} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} E_1 \\ E_2 \\ E_3 \end{Bmatrix}, \tag{18}$$

$$\begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & e_{15} & 0 \\ 0 & 0 & 0 & e_{15} & 0 & 0 \\ e_{13} & e_{13} & e_{33} & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \\ 2\varepsilon_{12} \end{Bmatrix} + \begin{bmatrix} \chi_{11} & 0 & 0 \\ 0 & \chi_{11} & 0 \\ 0 & 0 & \chi_{33} \end{bmatrix} \begin{Bmatrix} E_1 \\ E_2 \\ E_3 \end{Bmatrix}, \tag{19}$$

where $E_i = -\phi_{,i}$ are the components of the electric field. Considering the generalized plane deformation, with the poling direction parallel to x_3 where all stress and displacement components depend on (x_1, x_3) only, as shown in Fig. 1, the material matrices $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{T}$ can be written in the following way:

$$\mathbf{Q} = \begin{bmatrix} c_{11} & 0 & 0 & 0 \\ 0 & c_{44} & 0 & e_{15} \\ 0 & 0 & \frac{c_{11}-c_{12}}{2} & 0 \\ 0 & e_{15} & 0 & -\chi_{11} \end{bmatrix}, \tag{20}$$

$$\mathbf{R} = \begin{bmatrix} 0 & c_{13} & 0 & e_{13} \\ c_{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ e_{15} & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} c_{44} & 0 & 0 & 0 \\ 0 & c_{33} & 0 & e_{33} \\ 0 & 0 & c_{44} & 0 \\ 0 & e_{33} & 0 & -\chi_{33} \end{bmatrix}. \tag{21}$$

The construction of these matrices conforms to the Stroh formalism as described in Section 2, but with x_1 and x_3 as the in-plane coordinate variables (x_1 is the coordinate in the crack direction, and x_3 is in the poling direction) instead of x_1 and x_2. The material behavior is determined with the elastic properties as confined

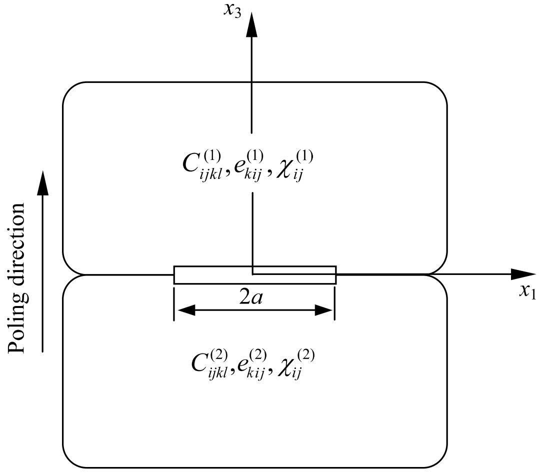

Fig. 1. The interface crack in a piezoelectric bimaterial.

by the matrices **Q**, **R**, and **T**, but not the choice of the coordinate system. Also, it should be pointed out in this case there is no apparent decoupling between the in-plane problem and the anti-plane problem, as oppose to the case when the poling axis is parallel to the crack front. Therefore, the following discussion should be generally applicable to mix mode problems of an interfacial crack.

Substituting Eqs. (20) and (21) into Eq. (4), the characteristic equation becomes

$$\begin{vmatrix} c_{11}+p^2c_{44} & p(c_{13}+c_{44}) & 0 & p(e_{13}+e_{15}) \\ p(c_{13}+c_{44}) & c_{44}+p^2c_{33} & 0 & e_{15}+p^2e_{33} \\ 0 & 0 & \frac{c_{11}-c_{12}}{2}+p^2c_{44} & 0 \\ p(e_{13}+e_{15}) & e_{15}+p^2e_{33} & 0 & -\chi_{11}-p^2\chi_{33} \end{vmatrix} = 0. \tag{22}$$

The explicit expression of all the characteristic roots of Eq. (22) have been obtained by Ou and Chen (2003). Here, we shall pay attention to one special characteristic root and its associated characteristic vector, denoted by p_3 and $\mathbf{a}_3$, as

$$p_3 = \mathrm{i}\sqrt{\frac{c_{11}-c_{12}}{2c_{44}}}, \quad \mathbf{a}_3 = [0 \quad 0 \quad a_{33} \quad 0]^{\mathrm{T}}, \tag{23}$$

since the others do not matter in the present problem. It should be noted that for some piezoelectric materials of class $3m$, the characteristic root is not the same as the one expressed in Eq. (23), and therefore the applicability of the present derivation is excluded for those materials.

Then, we can write the material matrices, **A** and **B**, according to Eqs. (6), (7) and (23), in the following forms:

$$\mathbf{A} = \begin{bmatrix} * & * & 0 & * \\ * & * & 0 & * \\ 0 & 0 & a_{33} & 0 \\ * & * & 0 & * \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} * & * & 0 & * \\ * & * & 0 & * \\ 0 & 0 & b_{33} & 0 \\ * & * & 0 & * \end{bmatrix}, \quad \mathbf{B}^{-1} = \begin{bmatrix} * & * & 0 & * \\ * & * & 0 & * \\ 0 & 0 & 1/b_{33} & 0 \\ * & * & 0 & * \end{bmatrix}, \tag{24}$$

where

$$b_{33} = \mathrm{i}\sqrt{(c_{11}-c_{12})/2c_{44}} \cdot c_{44} \cdot a_{33}, \tag{25}$$

and "$*$" presents other arbitrary complex numbers that do not play an role in the subsequent derivation.

Substituting Eqs. (24) and (25) into the first formula in Eq. (8), it yields

$$\mathbf{Y} = \begin{bmatrix} * & * & 0 & * \\ * & * & 0 & * \\ 0 & 0 & i\frac{a_{33}}{b_{33}} & 0 \\ * & * & 0 & * \end{bmatrix} = \begin{bmatrix} * & * & 0 & * \\ * & * & 0 & * \\ 0 & 0 & \sqrt{\frac{2c_{44}}{c_{11}-c_{12}}} & 0 \\ * & * & 0 & * \end{bmatrix} = \begin{bmatrix} * & * & 0 & * \\ * & * & 0 & * \\ 0 & 0 & \alpha & 0 \\ * & * & 0 & * \end{bmatrix}. \tag{26}$$

Clearly, α is a real number when $c_{11}-c_{12} \neq 0$, which is true for almost all transversely isotropic piezoelectric materials in the practical use. Thus we obtained, from Eqs. (8), (13) and (26), the following result:

$$\mathbf{W} = \mathrm{Im}\mathbf{H} = \mathrm{Im}\mathbf{Y}_1 + \mathrm{Im}\overline{\mathbf{Y}}_2 = \frac{1}{2\mathrm{i}}[(\mathbf{Y}_1 - \overline{\mathbf{Y}}_1) - (\mathbf{Y}_2 - \overline{\mathbf{Y}}_2)] = \begin{bmatrix} * & * & 0 & * \\ * & * & 0 & * \\ 0 & 0 & 0 & 0 \\ * & * & 0 & * \end{bmatrix}, \tag{27}$$

and thus the condition in Eq. (17) is proven to be satisfied for all transversely isotropic materials. This is indeed the case for the many practical piezoelectric materials evaluated numerically as shown in the later section.

When the condition in Eq. (17) is true, Eq. (14) reduces to

$$\beta^4 + 2b\beta^2 = 0 \tag{28}$$

and its roots can be found as

$$\beta_{1,2} = 0, \quad \beta_{3,4} = \pm\sqrt{-2b} = \pm\frac{1}{\sqrt{2}}\sqrt{-tr[(D^{-1}W)^2]}, \tag{29}$$

and hence

$$\gamma_{1,2} = 0, \quad \gamma_{3,4} = \pm\frac{1}{\pi}\tanh^{-1}[\sqrt{-2b}] = \pm\frac{1}{\pi}\tanh^{-1}\left[\frac{1}{\sqrt{2}}\sqrt{-tr[(D^{-1}W)^2]}\right]. \tag{30}$$

It is thus shown that the two non-zero roots, γ_3, γ_4 are either real or imaginary numbers, depending on the sign (positive or negative) of the parameter b, but there cannot be coexistence of real and imaginary roots in the system. That is, if speaking with the terminology of Suo et al. (1992), the condition that either $\varepsilon = 0$ or $\kappa = 0$ must be true for a given transversely isotropic piezoelectric bimaterial. Because the degeneration of Eq. (14) to Eq. (28). It is more straightforward to obtain the eigenroots γ from Eq. (30) for transversely isotropic piezoelectric bimaterials than Eq. (16). In the latter case, care should be exercised when $\sqrt{b^2}$ is performed when $c = 0$. Therefore, for transversely isotropic piezoelectric bimaterials, the parameters ε and κ can be redefined as

$$\varepsilon = |\mathrm{Re}(\gamma_{3,4})|, \quad \kappa = |\mathrm{Im}(\gamma_{3,4})| \tag{31}$$

Numerical evaluations were conducted for 15 piezoelectric bimaterial systems paired with six basic transversely isotropic piezoelectric bimaterials: PZT-4, $BaTiO_3$, PZT-5H, PZT-6B, PZT-7A, and PZT-7, in different combinations. The material constants of these materials are listed in Table 1 (Park and Sun, 1995a; Xiao et al., 2001; Wang, 1992; Dunn and Taya, 1994; Shindo et al., 2000). The values of ε and κ were calculated using Eq. (30) and the results are given in Tables 2 and 3.

Table 1
Material constants for some piezoelectric ceramics

		PZT-4	PZT-5H	PZT-6B	PZT-7A	PZT-7	$BaTiO_3$
c_{11}	10^{10} N m^{-2}	13.9	12.6	16.8	14.8	13.0	15.0
c_{12}	10^{10} N m^{-2}	7.78	5.50	6.00	7.62	8.30	6.60
c_{13}	10^{10} N m^{-2}	7.43	5.30	6.00	7.42	8.30	6.60
c_{33}	10^{10} N m^{-2}	11.3	11.7	16.3	13.1	11.9	14.6
c_{44}	10^{10} N m^{-2}	2.56	3.53	2.71	2.54	2.50	4.4
e_{13}	C m^{-2}	−6.98	−6.50	−0.90	−2.10	−10.3	−4.35
e_{33}	C m^{-2}	13.8	23.3	7.10	9.50	14.7	17.5
e_{15}	C m^{-2}	13.4	17.0	4.60	9.70	13.5	11.4
x_{11}	10^{-10} C (V m)$^{-1}$	60.0	151	36.0	81.1	171	98.7
x_{33}	10^{-10} C (V m)$^{-1}$	54.7	130	34.0	73.5	186	112

Table 2
Values of κ for some piezoelectric bimaterials

κ	PZT-4	$BaTiO_3$	PZT-5H	PZT-6B	PZT-7A	P-7
PZT-4	–	0.0508	0.0442	0.0168	0.0247	0.0367
$BaTiO_3$		–	0	0.0095	0.0206	0.0162
PZT-5H			–	0	0	0.0035
PZT-6B				–	0	0
PZT-7A					–	0.0023

Table 3
Values of ε for some piezoelectric materials

ε	PZT-4	$BaTiO_3$	PZT-5H	PZT-6B	PZT-7A	P-7
PZT-4	–	0	0	0	0	0
$BaTiO_3$		–	0.0130	0	0	0
PZT-5H			–	0.0219	0.0069	0
PZT-6B				–	0.0055	0.0121
PZT-7A					–	0

4. Discussion

Transversely isotropic piezoelectric bimaterial systems have wide applications in sensor and active control technologies. They do exhibit various crack growth behaviors (Pak, 1990, 1992; Suo et al., 1992; Sosa, 1991; Park and Sun, 1995a,b; Boem and Atluri, 1996; Deng and Meguid, 1998; McMeeking, 1999, 2001; Ma and Chen, 2001), which have not been completely understood. In this paper, this class of piezoelectric materials, which is perhaps the class of the most practical significance, is studied as a special case of the generalized anisotropic piezoelectric materials, particularly with $\det(\mathbf{W}) = 0$. Because the intrinsic properties of the material matrices, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{T}$, interestingly, some materials, e.g. PZT-4, when paired with others to form a bimaterial system never exhibit an oscillatory character in the singular stress field at the interfacial crack-tip, while the others do or do not, depending on with whom they pair. What particular properties do cause the different behavior is an intriguing question, which is not easy to answer at present. In the light of the above discussion, however, the piezoelectric bimaterial systems can be classified into two groups: (A) with $\varepsilon = 0$ which can be called κ-class piezoelectric bimaterials, and (B) with $\kappa = 0$ which called ε-class piezoelectric bimaterials, as listed in Table 4. This classification is not just superficial, since whether the singular stress field has an oscillatory component in the stress field at the crack-tip on the interface does affect the energy release rate, and hence the behavior of crack extension under a far-field mechanical–electrical load.

The present study proves that there always exist two eigenvalues having zero values in the problem of Eq. (12) for transversely isotropic piezoelectric bimaterials. Then, the question is that are their eigenvectors necessarily linearly independent. This means new solution needs to be sought for the problem, since the general solution given by Suo et al. (1992) and Boem and Atluri (1996) is based on non-zero eigenvalues. Currently, the authors are able to obtain linearly independent eigenvectors numerically for the selected bimaterial systems (see Appendix A). However, proof of that exist in all cases is not as straightforward as numerical evaluation. Therefore, it is suggested that researchers dealing with this class of piezoelectric materials should carefully check the material matrices $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{T}$ and be aware of the condition in Eq. (17) and seek special solution of the problem, Eq. (12), individually for the material system of their interest.

In numerical evaluations, the orthogonal properties of the material matrices $\mathbf{A}$ and $\mathbf{B}$ should be checked: i.e., (Ting, 1986, 1990; Deng and Meguid, 1998)

$$\begin{aligned} &\mathbf{A}^{\mathrm{T}}\mathbf{B} + \mathbf{B}^{\mathrm{T}}\mathbf{A} = \mathbf{I} = \overline{\mathbf{A}}^{\mathrm{T}}\overline{\mathbf{B}} + \overline{\mathbf{B}}^{\mathrm{T}}\overline{\mathbf{A}}, \quad \mathbf{A}^{\mathrm{T}}\overline{\mathbf{B}} + \mathbf{B}^{\mathrm{T}}\overline{\mathbf{A}} = \mathbf{0} = \overline{\mathbf{B}}^{\mathrm{T}}\mathbf{A} + \overline{\mathbf{A}}^{\mathrm{T}}\mathbf{B}, \\ &\mathbf{A}\mathbf{A}^{\mathrm{T}} + \overline{\mathbf{A}\mathbf{A}}^{\mathrm{T}} = \mathbf{0} = \mathbf{B}\mathbf{B}^{\mathrm{T}} + \overline{\mathbf{B}\mathbf{B}}^{\mathrm{T}}, \quad \mathbf{B}\mathbf{A}^{\mathrm{T}} + \overline{\mathbf{B}\mathbf{A}}^{\mathrm{T}} = \mathbf{I} = \mathbf{A}\mathbf{B}^{\mathrm{T}} + \overline{\mathbf{A}\mathbf{B}}^{\mathrm{T}}, \end{aligned} \tag{32}$$

Table 4
Classification of transversely isotropic piezoelectric bimaterials

Classification	Crack-tip stress feature	Piezoelectric bimaterials
κ-Class	Non-oscillating ($\varepsilon = 0$)	PZT-4/$BaTiO_3$, PZT-4/PZT-5H, PZT-4/PZT-6B, PZT-4/PZT-7A, PZT-4/P-7, $BaTiO_3$/PZT-6B, $BaTiO_3$/PZT-7A, $BaTiO_3$/P-7, PZT-5H/P-7, PZT-7A/P-7
ε-Class	Oscillating ($\kappa = 0$)	$BaTiO_3$/PZT-5H, PZT-5H/PZT-6B, PZT-5H/PZT-7A, PZT-6B/PZT-7A, PZT-5H/P-7

as it has been done in our numerical evaluation. These properties are important to ensure the correct solution. The numerical computation also shown that the values of ε and κ for transversely isotropic piezoelectric bimaterials are invariant with the choice of the dimension system of the physical quantities under consideration such as stress and electric field.

5. Conclusions

In summary, the following conclusions can be drawn from the present study.

(1) It is theoretically proven that the determinant of the imaginary part of the Hermitian matrix **H** vanishes for all transversely isotropic piezoelectric bimaterials, leading to a degenerated characteristic equation, Eq. (28), for the system. As a consequence, there always exist two zero eigenvalues for the problem with a crack present at the bimaterial interface. The other two eigenvalues, being either real or imaginary, should be evaluated from Eq. (30).
(2) If translated into Suo et al.'s terminology, the above point can be rephrased as that there is no coexistence of ε and κ for any given transversely isotropic piezoelectric bimaterial system. This means that such a material may or may not exhibit oscillatory singularity in the crack-tip stress and electrical displacement fields at the interface, depending on the pair of the materials. This intriguing phenomenon is due to the intrinsic material properties of the material matrices **Q**, **R**, and **T**.
(3) Numerical evaluations for 15 bimaterial systems consisting of commercially available piezoelectric ceramics such as PZT-4, PZT-5H, PZT-6B, PZT-7A, P-7 and $BaTiO_3$ have been performed and the results corroborates with the theoretical derivation.
(4) A classification is made for the transversely isotropic piezoelectric bimaterial systems, according to the criteria of whether $\varepsilon = 0$, or $\kappa = 0$. Henceforth, their fracture behaviors should be examined carefully with regards to the characteristics of the crack-tip fields in these bimaterial systems.
(5) On a numerical basis, linearly independent eigenvectors for the vanishing γ can still be obtained, which then complete the solution of Eq. (12). It can be seen that there always be four real linear independent basic vectors $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$, and $\mathbf{w}_4$ for the κ-class piezoelectric bimaterials rather than those described by Suo et al. (1992).

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Appendix A

Eigenvectors $\mathbf{w} = [\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4]$ for some piezoelectric bimaterials.

A.1. κ-Class piezoelectric bimaterials

(1) PZT-4/$BaTiO_3$:

$$\kappa_1 = 0.0508, \quad \kappa_2 = -0.0508, \quad \varepsilon_1 = \varepsilon_2 = 0,$$

$$\mathbf{w} = \begin{bmatrix} 0.9138 & 0.9138 & 0 & 0 \\ -0.0531 & 0.0531 & 0.9837 & 0 \\ 0 & 0 & 0 & 1 \\ 0.4027 & -0.4027 & 0.1798 & 0 \end{bmatrix};$$

(2) PZT-4/PZT-5H:

$$\kappa_1 = 0.0442, \quad \kappa_2 = -0.0442, \quad \varepsilon_1 = \varepsilon_2 = 0,$$

$$\mathbf{w} = \begin{bmatrix} 0.8535 & 0.8535 & 0 & 0 \\ -0.3459 & 0.3459 & 0.9995 & 0 \\ 0 & 0 & 0 & 1 \\ 0.3896 & -0.3896 & 0.0312 & 0 \end{bmatrix};$$

(3) PZT-4/PZT-6B:

$$\kappa_1 = 0.0168, \quad \kappa_2 = -0.0168, \quad \varepsilon_1 = \varepsilon_2 = 0,$$

$$\mathbf{w} = \begin{bmatrix} 0.7066 & 0.7066 & 0 & 0 \\ 0.5993 & -0.5993 & 0.9455 & 0 \\ 0 & 0 & 0 & 1 \\ 0.3763 & -0.3763 & 0.3256 & 0 \end{bmatrix};$$

(4) PZT-4/PZT-7A:

$$\kappa_1 = 0.0247, \quad \kappa_2 = -0.0247, \quad \varepsilon_1 = \varepsilon_2 = 0,$$

$$\mathbf{w} = \begin{bmatrix} 0.8940 & 0.8940 & 0 & 0 \\ 0.1296 & -0.1296 & 0.9646 & 0 \\ 0 & 0 & 0 & 1 \\ 0.4288 & -0.4288 & 0.2636 & 0 \end{bmatrix};$$

(5) PZT-4/P-7:

$$\kappa_1 = 0.0367, \quad \kappa_2 = -0.0367, \quad \varepsilon_1 = \varepsilon_2 = 0,$$

$$\mathbf{w} = \begin{bmatrix} 0.8748 & 0.8748 & 0 & 0 \\ -0.1721 & 0.1721 & 0.9906 & 0 \\ 0 & 0 & 0 & 1 \\ 0.4528 & -0.4528 & 0.1364 & 0 \end{bmatrix};$$

(6) $BaTiO_3$/PZT-6B:

$$\kappa_1 = 0.0095, \quad \kappa_2 = -0.0095, \quad \varepsilon_1 = \varepsilon_2 = 0,$$

$$\mathbf{w} = \begin{bmatrix} 0.7279 & 0.7279 & 0 & 0 \\ 0.6385 & -0.6385 & 0.9928 & 0 \\ 0 & 0 & 0 & 1 \\ -0.2500 & 0.2500 & -0.1196 & 0 \end{bmatrix};$$

(7) $BaTiO_3$/PZT-7A:

$$\kappa_1 = 0.0206, \quad \kappa_2 = -0.0206, \quad \varepsilon_1 = \varepsilon_2 = 0,$$

$$\mathbf{w} = \begin{bmatrix} 0.9058 & 0.9058 & 0 & 0 \\ 0.1726 & -0.1726 & 0.9978 & 0 \\ 0 & 0 & 0 & 1 \\ -0.3869 & 0.3869 & 0.0665 & 0 \end{bmatrix};$$

(8) $BaTiO_3$/P-7:

$$\kappa_1 = 0.0162, \quad \kappa_2 = -0.0162, \quad \varepsilon_1 = \varepsilon_2 = 0,$$

$$\mathbf{w} = \begin{bmatrix} 0.8504 & 0.8504 & 0 & 0 \\ -0.1070 & 0.1070 & 0.9587 & 0 \\ 0 & 0 & 0 & 1 \\ -0.5152 & 0.5152 & 0.2844 & 0 \end{bmatrix};$$

(9) PZT-5H/PZT-7:

$$\kappa_1 = 0.0035, \quad \kappa_2 = -0.0035, \quad \varepsilon_1 = \varepsilon_2 = 0,$$

$$\mathbf{w} = \begin{bmatrix} -0.3031 & 0.3031 & 0 & 0 \\ 0.8604 & 0.8604 & 0.9280 & 0 \\ 0 & 0 & 0 & 1 \\ -0.4096 & -0.4096 & -0.3725 & 0 \end{bmatrix};$$

(10) PZT-7A/P-7:

$$\kappa_1 = 0.0023, \quad \kappa_2 = -0.0023, \quad \varepsilon_1 = \varepsilon_2 = 0,$$

$$\mathbf{w} = \begin{bmatrix} -0.3020 & 0.3020 & 0 & 0 \\ 0.8810 & 0.8810 & 0.9426 & 0 \\ 0 & 0 & 0 & 1 \\ -0.3632 & -0.3632 & -0.3340 & 0 \end{bmatrix}.$$

A.2. ε-Class bimaterials

(1) $BaTiO_3$/PZT-5H:

$$\varepsilon_1 = 0.0130, \quad \varepsilon_2 = -0.0130, \quad \kappa_1 = \kappa_2 = 0,$$

$$\mathbf{w} = \begin{bmatrix} -0.6480\mathrm{i} & 0.6480\mathrm{i} & 0 & 0 \\ 0.6944 & 0.6944 & 0.5683 & 0 \\ 0 & 0 & 0 & 1 \\ 0.3128 & 0.3128 & 0.8228 & 0 \end{bmatrix};$$

(2) PZT-5H/PZT-6B:

$$\varepsilon_1 = 0.0219, \quad \varepsilon_2 = -0.0219, \quad \kappa_1 = \kappa_2 = 0,$$

$$\mathbf{w} = \begin{bmatrix} 0.6832\mathrm{i} & -0.6832\mathrm{i} & 0 & 0 \\ 0.7300 & 0.7300 & 0.7743 & 0 \\ 0 & 0 & 0 & 1 \\ -0.0203 & -0.0203 & -0.6328 & 0 \end{bmatrix};$$

(3) PZT-5H/PZT-7A:

$$\varepsilon_1 = 0.0069, \quad \varepsilon_2 = -0.0069, \quad \kappa_1 = \kappa_2 = 0,$$

$$\mathbf{w} = \begin{bmatrix} 0.3569\mathrm{i} & -0.3569\mathrm{i} & 0 & 0 \\ 0.8956 & 0.8956 & 0.9350 & 0 \\ 0 & 0 & 0 & 1 \\ -0.2654 & -0.2654 & -0.3548 & 0 \end{bmatrix};$$

(4) PZT-6B/PZT-7A:

$$\varepsilon_1 = 0.0055, \quad \varepsilon_2 = -0.0055, \quad \kappa_1 = \kappa_2 = 0,$$

$$\mathbf{w} = \begin{bmatrix} -0.6561\mathrm{i} & 0.6561\mathrm{i} & 0 & 0 \\ 0.7316 & 0.7316 & 0.7861 & 0 \\ 0 & 0 & 0 & 1 \\ 0.1853 & 0.1853 & 0.6181 & 0 \end{bmatrix};$$

(5) PZT-6B/P-7:

$$\varepsilon_1 = 0.0121, \quad \varepsilon_2 = -0.0121, \quad \kappa_1 = \kappa_2 = 0,$$

$$\mathbf{w} = \begin{bmatrix} 0.7116 & 0.7116 & 0 & 0 \\ 0.7021\mathrm{i} & -0.7021\mathrm{i} & -0.4550 & 0 \\ 0 & 0 & 0 & 1 \\ 0.0236\mathrm{i} & -0.0236\mathrm{i} & 0.8905 & 0 \end{bmatrix}.$$

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